HW 4 P70

(1a) $f(z) = x - iy = u + iv$

$$\begin{cases} u_x = 1, & v_y = -1 \\ u_y = 0 & v_x = 0 \end{cases} \Rightarrow f' \text{ DNE}$$

(b) $f(z) = z - \bar{z} = 2yi = u + iv$

$$\begin{cases} u_x = 0, & v_y = 2 \\ u_y = 0 & v_x = 0 \end{cases} \Rightarrow f' \text{ DNE}$$

(c) $f(z) = 2x + ixy^2 = u + iV$

$$\begin{cases} u_x = 2, & v_y = 2xy \\ u_y = 0 & v_x = y^2 \end{cases}$$

If $u_y = -v_x$, then $y = 0$, But $u_x \neq v_y$ if $y = 0$, Thus f' DNE.

(d) $f(z) = e^x e^{-iy} = e^x(\cos x - i\sin y)$

$$\begin{cases} u_x = e^x \cos y & v_y = -e^x \cos y \\ u_y = -e^x \sin y, & v_x = -e^x \sin y \end{cases}$$

Similar to (1c), f' DNE.

(2a) $f = 2 - y + ix = u + iv$

$$\begin{cases} u_x = 0 = v_y \\ u_y = -1 = -v_x \end{cases} \Rightarrow f' \text{ exists and } f' = i, \text{ then } f''$$

exists everywhere clearly.

2b $f = e^{-x}(\cos y - \sin y)$

$\begin{cases} u_x = -e^{-x}\cos y = v_y \\ u_y = -e^{-x}\sin y = -v_x \end{cases} \Rightarrow f'$ exists and $f' = -e^{-x}(\cos y - \sin y)$

let $f' = p + qi$, then $\begin{cases} p_x = e^{-x}\cos y = q_y \\ p_y = e^{-x}\sin y = -q_x \end{cases} \Rightarrow f''$ exists.

and $f'' = e^{-x}\cos y - ie^{-x}\sin y = f$.

2c $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$

$\begin{cases} u_x = 3x^2 - 3y^2 = v_y \\ u_y = -6xy = -v_x \end{cases} \Rightarrow f'$ exists and $f' = (3x^2 - 3y^2) + 6xyi$

If $f' = p + qi$, then $\begin{cases} p_x = 6x = q_y \\ p_y = -6y = -q_x \end{cases} \Rightarrow f''$ exists and

$f'' = 6x + 6yi = 6z$.

2d $\begin{cases} u_x = -\sin x\cosh y = v_y \\ u_y = \cos x\sinh y = -v_x \end{cases} \Rightarrow f'$ exists and

$f' = -\sin x\cosh y - i\cos x\sinh y = p + qi$

$\begin{cases} p_x = -\cos x\cosh y = q_y \\ p_y = -\sin x\sinh y = -q_x \end{cases} \Rightarrow f''$ exists and

$f'' = -\cos x\cosh y + i\sin x\sinh y = -f$.

3a $f(z) = \frac{1}{z} = \frac{x - iy}{x^2+y^2}$

$u_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$, $v_y = -\left(\frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}\right)$

$= \frac{y^2 - x^2}{(x^2+y^2)^2}$ $\qquad = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\Rightarrow u_x = v_y$ if $z \neq 0$.

$u_y = \frac{\cancel{(x^2+y^2)} - x(2y)}{(x^2+y^2)^2}$ $\qquad v_x = \frac{+y(2x)}{(x^2+y^2)^2}$

$\Rightarrow u_y = -v_x$ if $z \neq 0$.

$f'(z) = -1/z^2$.

b $\begin{cases} u_x = 2x, & v_y = 2y \\ u_y = 0, & v_x = 0 \end{cases}$ $\Rightarrow f'$ exists at $x=y$, $f' = 2x$

c $f(z) = z\,\mathrm{Im}(z) = xy + iy^2$

$\begin{cases} u_x = y, & v_y = 2y \\ u_y = x, & v_x = 0 \end{cases}$ $\Rightarrow f'$ exists only at $z = 0$

and $f'(0) = \lim_{z \to 0} \frac{z\,\mathrm{Im}(z)}{z} = 0$

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1a $\begin{cases} u_x = 3, & v_y = 3 \\ u_y = 1 & v_x = -1 \end{cases}$ $\Rightarrow f$ is entire.

(1b) $\begin{cases} u_x = \sinh x \cos y, & v_y = \sinh x \cos y \\ u_y = -\cosh x \sin y, & v_x = \cosh x \sin y \end{cases} \Rightarrow f$ is entire.

(1c) $\begin{cases} u_x = e^{-y}\cos x, & v_y = e^{-y}\cos x \\ u_y = -e^{-y}\sin x & v_x = e^{-y}\sin x \end{cases} \Rightarrow f$ is ~~[illegible]~~ entire

(1d) $f = (x^2-y^2+2xyi-2)\, e^{-x}(\cos y - i\sin y)$

$= e^{-x}\left((x^2-y^2-2)\cos y + 2xy\sin y + i\left(2xy\cos y - (x^2-y^2-2)\sin y\right)\right)$

$u_x = -e^{-x}\left((x^2-y^2-2)\cos y + 2xy\sin y\right) + e^{-x}(2x\cos y + 2y\sin y)$

$= e^{-x}\left((-x^2+2x+y^2+2)\cos y + 2y\sin y(-x+1)\right)$

$v_y = e^{-x}\left(2x(\cos y - y\sin y) - (x^2-y^2-2)\cos y + 2y\sin y\right)$

$= e^{-x}\left((-x^2+2x+y^2+2)\cos y + 2y\sin y(1-x)\right)$

$u_y = e^{-x}\left(-(x^2-y^2-2)\sin y - 2y\cos y + 2x\sin y + 2xy\cos y\right)$

$= e^{-x}\left((2x-x^2+y^2+2)\sin y + (2xy-2y)\cos y\right)$

$= \cancel{V_x} - V_x \Rightarrow f$ is entire.

(2a) $\begin{cases} u_x = y, & v_y = 1 \\ u_y = x, & v_x = 0 \end{cases} \Rightarrow$ It's diff only at $z = i$,

Thus it is nowhere analytic.

(2b) $\begin{cases} u_x = 2y, & v_y = -2y \\ u_y = 2x, & v_x = 2x \end{cases} \Rightarrow$ It is diff only at $z=0$

It is nowhere analytic.

2c $f(z) = e^y(\cos x + i\sin x)$

$$\begin{cases} u_x = -e^y \sin x, & v_y = e^y \sin x \\ u_y = e^y \cos x, & v_x = e^y \cos x \end{cases}$$

It's nowhere diff. hence nowhere analytic.

7 If f is real-valued, then $v=0$ and $u_x = u_y = 0$.
Since $u \in C^1(D)$, thus u = constant.

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(2a) $\int_0^1 (1+it)^2 dt = \int_0^1 1-t^2 dt + i\int_0^1 2t\, dt$

$= \frac{2}{3} + i$

(2b) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt = \int_0^1 \frac{1}{t^2} - i\, dt - i\int_1^2 \frac{2}{t} dt$

$= -1/2 - i \log 4$

(2c) $\int_0^{\pi/6} e^{2ti} = \frac{\sqrt{3}}{4} + \frac{i}{4}$

(2d) $\int_0^{\infty} e^{-zt} dt = \left[\frac{e^{-zt}}{-z}\right]_0^{\infty} = \frac{1}{z}$.

(3) If $m=n$, $\int_0^{2\pi} d\theta = 2\pi$

If $m \neq n$, $\int_0^{2\pi} e^{i\theta(m-n)} d\theta = \left[\frac{1}{i(m-n)} e^{i(m-n)\theta}\right]_0^{2\pi}$

$= \frac{1}{i(m-n)} (1-1) = 0$

(4) $\int_0^{\pi} e^{(1+i)x} dx = \frac{1}{1+i} \left(e^{(1+i)\pi} - 1\right)$

Comparing real and imaginary part,

$\int_0^{\pi} e^{t} \cos x = -\frac{1}{2}(1+e^{\pi})$, $\int_0^{\pi} e^{x} \sin x = \frac{1}{2}(1+e^{\pi})$

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① a $dz = 2ie^{i\theta}d\theta$

$$\int_C f(z)\,dz = \int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}} \cdot 2ie^{i\theta}\,d\theta$$

$$= 2i\int_0^{\pi} e^{i\theta}+1$$

$$= -4+2\pi i$$

1b $$\int_C f(z)\,dz = \int_{\pi}^{2\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}} \cdot 2ie^{i\theta}$$

$$= 4+2\pi i$$

1c $$\int_C f(z)\,dz = \int_0^{\pi} f(z)\,2ie^{i\theta}\,d\theta + \int_{\pi}^{2\pi} f(z)\,2ie^{i\theta}\,d\theta$$

$$= 4\pi i$$

~~2a $z = 2e^{i\theta}$, $\theta \in [0, \pi/2$~~

2a $dz = ie^{i\theta}d\theta$

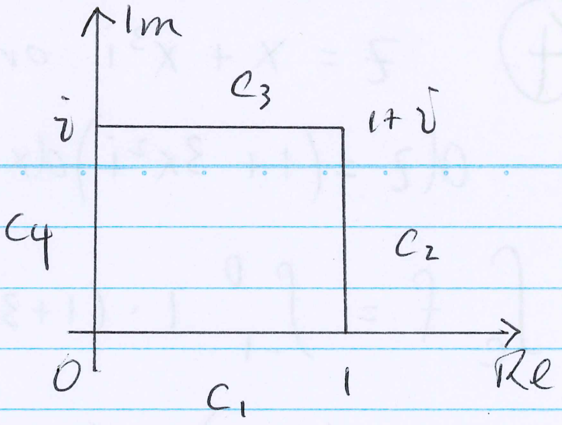
$$\int_C z-1 = \int_{\pi}^{2\pi} e^{i\theta} \cdot ie^{i\theta}\,d\theta$$

$$= \frac{i}{2i}\left[e^{2i\theta}\right]_{\pi}^{2\pi} = 0$$

2b $dz = dx$,

$$\int_C z-1 = \int_0^2 x-1 = 0$$

③ In C_1, $z = x$



$$\int_{C_1} f\,dz = \int_0^1 \pi e^{\pi x}\,dx$$

$$= (e^{\pi} - 1)$$

In C_2, $z = 1 + yi$

$$\int_{C_2} f\,dz = \int_0^1 \pi e^{\pi(1-yi)}\,i\,dy$$

$$= -e^{\pi}(e^{-\pi i} - 1) = 2e^{\pi}$$

In C_3, $z = x + i$

$$\int_{C_3} f\,dz = \int_1^0 \pi e^{\pi(x-i)}\,dx$$

$$= e^{-\pi i}(1 - e^{\pi}) = (e^{\pi} - 1)$$

In C_4, $z = yi$

$$\int_{C_4} f\,dz = \int_1^0 \pi e^{-\pi y i}\,i\,dy$$

$$= -(1 - e^{\pi i}) = -2$$

Summing up, $\int_C f\,dz = 4(e^{\pi} - 1)$

(4) $z = x + x^3 i$ on the curve $y = x^3$

$dz = (1 + 3x^2 i)\,dx$

$$\int_C f = \int_{-1}^{0} 1 \cdot (1+3x^2 i)\,dx + \int_0^1 4x^3(1+3x^2 i)\,dx$$

$$= (1+i) + (1+2i) = 2+3i$$

(5) Since the antiderivative of $f = 1$ is $F = z$,

(6) $dz = i e^{i\theta} d\theta$

$$\int_{-\pi}^{\pi} z^i\,dz = \int_{-\pi}^{\pi} e^{i(i\theta)} \cdot i e^{i\theta} d\theta$$

$$= \frac{i}{i-1}\left[e^{\theta(i-1)} \right]_0^{\pi}$$

$$= \frac{1-i}{2}\left(-e^{-\pi} - 1 \right).$$